Determination of the Correlation Spectrum of Oscillators with Low Noise

FRANZ X. KAERTNER

Abstract —**A general expression for the correlation spectrum of an oscillator, described by a set of nonlinear ordinary differential equations with intrinsic noise sources, is derived by a first-order perturbation theory. The analytical derivations are well suited to the numerical determination of the correlation spectrum by Poincaré mapping methods. The theory is applied to a lumped circuit model of a Colpitts oscillator. The noise behavior of complex oscillator circuits used in microwave engineering may be simulated by the derived method.**

I. Introduction

THE THEORIES presented in most papers on noise in oscillators allow a good qualitative understanding of the noise behavior of oscillators [1], [10] and give rules for the design of low-noise oscillators. But the procedures used for this approach are not suitable for calculating the noise spectra of oscillators on the basis of the large-signal models of active components with the various intrinsic noise sources of the active and passive components.

In this paper, a general time-domain analysis of noise in oscillators is presented. In Section II a precise definition of the problem treated here is given. Assumptions about the noise sources, which can be simulated with the presented formalism, are also outlined. In Section III the inherent stochastic processes of an oscillator are derived by a first-order perturbation theory. A general expression for the correlation matrix of an oscillator is derived in Section IV. In Section V we present numerical techniques for determining the complete correlation matrix or correlation spectrum of an oscillator signal. Finally in Section VI we will apply the formalism presented here to a Colpitts oscillator where the transistor is modeled by an Ebers–Moll equivalent circuit.

II. Outline of the Problem

A dynamical system is described by a set of N first-order nonlinear differential equations:

$$\dot{\vec{x}} = \vec{F}(\vec{x}, t, \vec{\xi}\,), \qquad \vec{x} \in \mathbb{R}^N; \vec{\xi} \in \mathbb{R}^K. \tag{1}$$

The components x_i of the vector $\vec{x}$ uniquely determine the state of the system. In electrical systems, these state variables are the linear independent voltages at the capacitors and the currents through the inductors. These variables determine the energy stored in the network [11]. The vector $\vec{\xi}$ describes the stochastic noise sources, which are always present in dissipative systems [12]. If the system equation (1) descirbes an oscillator without external synchronisation, the right-hand side of (1) does not explicitly depend on time; i.e., the system without the noise sources is autonomous. In electrical oscillators the noise sources are very small in comparison with the state variables. Thus it is sufficient to take into account the noise sources up to first order. Thus we obtain

$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{0}) + \boldsymbol{G}(\vec{x})\vec{\xi} \tag{2}$$

where $\boldsymbol{G}(\vec{x}) \in \mathbb{R}^N \times \mathbb{R}^K$ with elements

$$G_{ij}(\vec{x}) = \left.\frac{\partial F_i(\vec{x}, \vec{\xi}\,)}{\partial \xi_j}\right|_{\vec{\xi} = \vec{0}}. \tag{3}$$

Thus $\boldsymbol{G}(\vec{x})$ describes the influence of the noise sources on the state variables and also the modulation of the intensity of the noise sources by the state variables, as in the case of shot noise, where the intensity of noise is proportional to the current through the components. This will be discussed in detail in the example given in Section VI. Furthermore we shall assume that the noise sources given by $\vec{\xi}$ are white and Gaussian.

Thus the statistics of the noise sources are completely described by [13]

$$\langle \xi_i(t) \xi_j(t') \rangle = \Gamma_{ij} \delta(t - t') \tag{4}$$

$$\langle \xi_i(t) \rangle = 0 \tag{5}$$

$$p(\vec{\xi}\,) = \left((2\pi \cdot \Delta f)^K \det(\Gamma)\right)^{-1/2} \cdot \exp\left[-\frac{\vec{\xi}^{\dagger} \Gamma^{-1} \vec{\xi}}{2 \cdot \Delta f}\right] \tag{6}$$

Γ is the correlation matrix of the K-dimensional stationary noise process, and Δf is the bandwidth with which the probability distribution was measured. This restriction to white noise sources does not preclude simulating the influence of colored noise because we can extend our original system given by (2) by a linear system $\boldsymbol{L}$, which produces the colored noise out of the white sources:

$$\dot{\vec{x}} = \vec{F}(\vec{x}) + \boldsymbol{G}(\vec{x})\vec{y}$$
$$\dot{\vec{y}} = \boldsymbol{L}\vec{y} + \vec{\xi}. \tag{7}$$

Manuscript received April 14, 1987; revised April 13, 1988.
The author is with the Institut für Hochfrequenztechnik, Technische Universität München, 8000 München 2, West Germany.
IEEE Log Number 8824241.

0018-9480/89/0100-0090$01.00

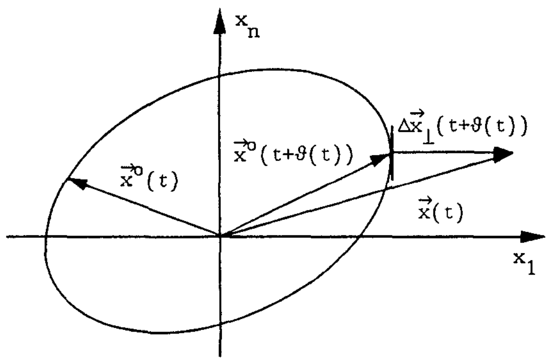

Fig. 1. Separation of the perturbed motion into normal and tangential deviations.

If we denote the vector $(\vec{x}, \vec{y})$ as the new state vector, we have a problem of the same kind as stated above.

Since the differential equation (1) describes an oscillator, the solution of the unperturbed system, setting $\vec{\xi}(t) = \vec{0}$, is the stable limit cycle $\vec{x}^0(t)$ with period T^0. The noise sources produce deviations from this stable limit cycle, broadening the oscillator's spectrum. In the following the complete spectrum of an oscillator is derived by interpreting the differential equation (2) as a stochastic Ito differential equation [14], [15]:

$$d\vec{x} = \vec{F}(\vec{x})\, dt + \boldsymbol{G}(\vec{x})\, d\vec{W}(t) \tag{8}$$

$$\langle d\vec{W}(t)\rangle = 0 \tag{9}$$

$$\langle d\vec{W}(t)\, d\vec{W}(t')\rangle = \Gamma_{ij}\delta_{t,t'}\, dt. \tag{10}$$

With these assumptions we will derive the stochastic processes for the amplitude and phase of an oscillator. These processes will be Markov processes because of the assumed whiteness of the noise sources.

III. Derivation of Amplitude and Phase Process

The noise sources $\vec{\xi}(t)$ force the oscillator signal $\vec{x}(t)$ to deviate from the limit cycle $\vec{x}^0(t)$, which is a closed orbit in the N-dimensional phase space of the possible states $\vec{x}$ of the system (see Fig. 1). Since the limit cycle is stable, the deviations perpendicular to the limit cycle may remain small. The deviations in the direction of the unperturbed orbit, however, may become unbounded in time, even though $\vec{x}^0(t)$ and $\vec{x}(t)$ have the same initial values. This is possible because the differential equation (2) is autonomous without the noise sources. To be able to use perturbation methods we separate the solution $\vec{x}(t)$ as shown in Fig. 1:

$$\vec{x}(t) = \vec{x}^0(t+\vartheta(t)) + \Delta\vec{x}_\perp(t+\vartheta(t)). \tag{11}$$

The stochastic variable $\vec{x}(t)$ can be separated in the part $\Delta\vec{x}_\perp(t+\vartheta(t))$, which is an element of the orthogonal complement space $\mathscr{N}(t)$ to the tangent space at the limit cycle at point $\vec{x}^0(t+\vartheta(t))$ and the time-shifted solution $\vec{x}^0(t+\vartheta(t))$ of the unperturbed equation [16]. The stochastic timeshift $\vartheta(t)$ leads to a random phase modulation in the unperturbed solution, as we will see later on. Therefore $\vartheta(t)$ determines the phase process of the oscillator and we will call $\Delta\vec{x}_\perp$ the amplitude process. By this subdivision of the signal the relation

$$\|\Delta\vec{x}_\perp\| \ll \|\vec{x}^0\| \tag{12}$$

is valid for every time t. With the statement given by (11) we can derive two separate differential equations for amplitude and phase processes. Inserting (11) into (2) and neglecting terms of higher order in $\Delta\vec{x}_\perp$, $\dot{\vartheta}$ and the noise sources $\vec{\xi}$, we obtain

$$\vec{x}^{0\prime}(y)\dot{\vartheta} + \Delta\vec{x}'_\perp(y) = \mathbf{DF}\left(\vec{x}^0(y)\right)\Delta\vec{x}_\perp(y) + \boldsymbol{G}\left(\vec{x}^0(y)\right)\vec{\xi}\,(t(y)) \tag{13}$$

with

$$\mathrm{DF}\left(\vec{x}^0(y)\right)_{ij} = \left.\frac{\partial F_i(\vec{x})}{\partial x_j}\right|_{\vec{x}=\vec{x}^0(y)} \tag{14}$$

and

$$y = t + \vartheta(t). \tag{15}$$

The bar denotes the derivation with respect to the argument. We define the tangent vector $\vec{n}(t)$ at the unperturbed orbit $\vec{x}^0(t)$ by

$$\vec{n}(t) = \frac{\dot{\vec{x}}^0(t)}{\|\dot{\vec{x}}^0(t)\|} \tag{16}$$

and the projection operator $\boldsymbol{P}(t)$ by

$$\boldsymbol{P}(t) = \mathbf{1} - \vec{n}(t)\cdot\vec{n}(t)^T. \tag{17}$$

$\boldsymbol{P}(t)$ projects onto the hyperplane orthogonal to the orbit $\vec{x}^0(t)$. From

$$\vec{n}(t)^T\cdot\Delta\vec{x}_\perp(t) = 0 \tag{18}$$

we obtain the identity

$$\dot{\vec{n}}(t)^T\Delta\vec{x}_\perp(t) = -\vec{n}(t)^T\Delta\dot{\vec{x}}_\perp(t). \tag{19}$$

By application of the projector $\boldsymbol{P}(y)$ to (13) and with (19) we obtain

$$\frac{d}{dy}\Delta\vec{x}_\perp(y) = \boldsymbol{V}(y)\,\Delta\vec{x}_\perp(y) + \boldsymbol{Q}(y)\vec{\xi}\,(t(y)) \tag{20}$$

for the amplitude process, with the abbreviations

$$\boldsymbol{V}(y) = \boldsymbol{P}(y)\mathbf{DF}\left(\vec{x}^0(y)\right) - \vec{n}(y)\vec{n}(y)'^T \tag{21a}$$

$$\boldsymbol{Q}(y) = \boldsymbol{P}(y)\boldsymbol{G}(y). \tag{21b}$$

From the scalar product of $\vec{n}(y)$ with (13), we obtain the equation describing the phase process:

$$\dot{\vartheta}(t) = \beta(y)\,\Delta\vec{x}_\perp(y) + \alpha(y)\vec{\xi}\,(t(y)) \tag{22}$$

with

$$\beta(y) = \frac{\vec{n}(y)^T\mathbf{DF}\left(\vec{x}^0(y)\right) + \vec{n}(y)^{T\prime}}{\|\vec{x}^{0\prime}(y)\|} \tag{23a}$$

$$\alpha(y) = \frac{\vec{n}(y)^T\boldsymbol{G}(y)}{\|\vec{x}^{0\prime}(y)\|}. \tag{23b}$$

From (20) it can be seen that the amplitude process is described by a linear inhomogeneous differential equation with periodic coefficients. Therefore we can write the amplitude process

$$\Delta\vec{x}_{\perp}(y)=\int_{-\infty}^{y}\boldsymbol{\Psi}(y,s)\boldsymbol{Q}(s)\vec{\xi}\,(t(s))\,ds \qquad (24)$$

where $\boldsymbol{\Psi}(y,s)$ is the fundamental matrix of the homogeneous differential equation [17]. The integral is taken from infinity to achieve independence of initial conditions. Since the limit cycle is stable, $N-1$ Floquet exponents of the fundamental matrix must have negative real parts. One of the Floquet exponents is $-\infty$ because of the projection operator involved in equation (20) (see the Appendix). The phase process described by (22) can be integrated directly. For this we substitute the differential dt for dy according to (15):

$$dy=\left(1+\dot{\vartheta}(t)\right)dt \qquad (25)$$

and obtain

$$\vartheta(t)=\int_{0}^{t+\vartheta(t)}\left[\boldsymbol{\beta}(y)\,\Delta\vec{x}_{\perp}(y)+\boldsymbol{\alpha}(y)\vec{\xi}\,(t(y))\right]dy. \qquad (26)$$

For the determination of $\vartheta(t)$ up to first order in the noise sources, we can neglect $\vartheta(t)$ at the upper integration range. The physical significance of the derived stochastic processes for amplitude and phase, described by (24) and (26), is well known. The amplitude process is a multivariate Ornstein–Uhlenbeck process with periodic coefficients [15], [18]. And the phase process is a diffusion process [15], [19], which is driven partly by the amplitude deviations via the amplitude phase conversion matrix $\boldsymbol{\beta}$ and partly by the noise sources.

IV. The Correlation Spectrum of an Oscillator Signal

The correlation matrix of a stationary stochastic process $\vec{x}(t)$ is defined as follows [20]:

$$\boldsymbol{C}_{xx}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}\langle\vec{x}(t+\tau)\vec{x}^{\dagger}(t)\rangle\,dt. \qquad (27)$$

If we put (11) into (27) we obtain for the correlation matrix

$$\boldsymbol{C}_{xx}(\tau)=\boldsymbol{C}_{x^0x^0}(\tau)+\boldsymbol{C}_{x^0\Delta x}(\tau)+\boldsymbol{C}_{\Delta xx^0}(\tau)+\boldsymbol{C}_{\Delta x\Delta x}(\tau) \qquad (28)$$

$$\boldsymbol{C}_{x^0x^0}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}\langle\vec{x}^{0}(t+\tau+\vartheta(t+\tau))\cdot\vec{x}^{0\dagger}(t+\vartheta(t))\rangle\,dt \qquad (29)$$

$$\boldsymbol{C}_{x^0\Delta x}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}\langle\vec{x}^{0}(t+\tau+\vartheta(t+\tau))\,\vec{\Delta}x_{\perp}^{\dagger}\cdot(t+\vartheta(t))\rangle\,dt \qquad (30)$$

$$\boldsymbol{C}_{\Delta xx^0}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}\langle\Delta\vec{x}_{\perp}(t+\tau+\vartheta(t+\tau))\cdot\vec{x}^{0\dagger}(t+\vartheta(t))\rangle\,dt \qquad (31)$$

$$\boldsymbol{C}_{\Delta x\Delta x}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}\langle\Delta\vec{x}_{\perp}\cdot(t+\tau+\vartheta(t+\tau))\,\Delta\vec{x}_{\perp}^{\dagger}(t+\vartheta(t))\rangle\,dt. \qquad (32)$$

Equation (29) describes the phase noise, (30) and (31) describe the correlations between phase and amplitude noise, and (32) describes the amplitude noise. These parts of the correlation matrix will be subsequently calculated. Therefore we expand the limit cycle $\vec{x}^{0}(t)$ with period T^{0} into a Fourier series:

$$\vec{x}^{0}(t)=\sum_{n=-\infty}^{+\infty}\vec{A}_{n}e^{jn\omega_0 t},\qquad \text{with } \omega_0=\frac{2\pi}{T^0}. \qquad (33)$$

A. Determination of the Phase Noise

According to (29) the phase noise is given by

$$\boldsymbol{C}_{x^0x^0}(\tau)=\sum_{m,n=-\infty}^{+\infty}\vec{A}_{n}\vec{A}_{m}^{\dagger}\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}e^{j(n-m)\omega_0 t}\,dt\times e^{jn\omega_0\tau}\langle e^{j(n\omega_0\vartheta(t+\tau)-m\omega_0\vartheta(t))}\rangle. \qquad (34)$$

The ensemble average in (34) does not depend on time t, as we will see later on; therefore we can separate the time average and the ensemble average. The integral in (34) is a Kronecker symbol in the limit:

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}e^{j(n-m)\omega_0 t}\,dt=\delta_{n,m}. \qquad (35)$$

Thus we obtain

$$\boldsymbol{C}_{x^0x^0}(\tau)=\sum_{n=-\infty}^{+\infty}\vec{A}_{n}\vec{A}_{n}^{\dagger}e^{jn\omega_0\tau}\phi(n,\tau) \qquad (36)$$

where

$$\phi(n,\tau)=\langle e^{jn(\varphi(t+\tau)-\varphi(t))}\rangle \qquad (37)$$

and

$$\varphi(t)=\omega_0\vartheta(t). \qquad (38)$$

This definition of the phase helps us to avoid the often-used instantaneous frequency discussed in [21], [22], which has no physical significance.

The function $\phi(n,\tau)$ is the characteristic function of the stochastic variable $\Delta\varphi(t,\tau)$ describing the phase difference between $t+\tau$ and t [23], [24]:

$$\phi(n,\tau)=\langle e^{jn\Delta\varphi(t,\tau)}\rangle \qquad (39)$$

$$\Delta\varphi(t,\tau)=\varphi(t+\tau)-\varphi(t). \qquad (40)$$

From a knowledge of the characteristic function we can calculate the correlation matrix $\boldsymbol{C}_{x^0x^0}(\tau)$ according to (36). The corrlation spectrum due to phase noise can be obtained by Fourier transformation of the correlation matrix:

$$\boldsymbol{C}_{x^0x^0}(f)=\sum_{n=-\infty}^{+\infty}\vec{A}_{n}\vec{A}_{n}^{\dagger}F_{n}(f-nf_0) \qquad (41)$$

with

$$F_n(f) = \mathscr{F}\{\phi(n,\tau)\}. \tag{42}$$

Thus the Fourier transform of the characteristic function uniquely determines the spectral shape of the nth harmonic. For the calculation of the characteristic function we need the probability distribution of phase $\varphi(t)$. Since we have only assumed white Gaussian noise sources, the phase has to be a Gaussian distributed Markov process because of the linear dependence on the noise sources according to (20) and (22). Taking the case of non-Gaussian noise sources, the probability distribution of the phase becomes Gaussian in the course of time due to the central limit theorem. Therefore the assumption of a Gaussian distributed phase may be taken as an approximation in the case of non-Gaussian noise sources as well. Thus we can make the following statement for the conditional probability of the phase φ:

$$p(\varphi, t/\varphi_0, t_0) = (2\pi\sigma(\varphi_0, t-t_0))^{-1/2} \cdot \exp\left[-\frac{(\varphi-\varphi_0)^2}{2\sigma(\varphi_0, t-t_0)}\right] \tag{43a}$$

with

$$\sigma(\varphi_0, t-t_0) = \langle(\varphi(t)-\varphi(t_0))^2\rangle \tag{43b}$$

$$\varphi_0 = \varphi(t_0). \tag{43c}$$

The variance of the phase depends only on the time difference and the initial value of the Markov process on the limit cycle. Therefore the variance is periodic in φ_0 with a period of 2π because of the periodicity of the limit cycle. The joint probability distribution is given by

$$p(\varphi, t; \varphi_0, t_0) = p(\varphi, t/\varphi_0, t_0)\, p(\varphi_0, t_0). \tag{44}$$

With (39) and (44) we obtain

$$\phi(n,\tau) = \iint_{-\infty}^{+\infty} \{\exp(jn(\varphi_{t+\tau} - \varphi_t)) \cdot p(\varphi_{t+\tau}/\varphi_t)\, p(\varphi_t)\}\, d\varphi_{t+\tau}\, d\varphi_t. \tag{45}$$

Inserting eq. (43a) into (45) and replacing the phase difference $\varphi_{t+\tau} - \varphi_t$ with $\Delta\varphi$, we can do the integration about $\Delta\varphi$ and obtain

$$\phi(n,\tau) = \int_{-\infty}^{+\infty} p(\varphi_t) \exp\left(-\frac{n^2}{2}\sigma(\varphi_t, \tau)\right) d\varphi_t. \tag{46}$$

Since the variance is periodic in φ_t, we make the following Fourier expansion:

$$\sigma(\varphi,\tau) = \sum_{m=-\infty}^{+\infty} \sigma_m(\tau) e^{jm\varphi} \tag{47a}$$

with

$$\sigma_m(\tau) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(\varphi,\tau) e^{-jm\varphi}\, d\varphi. \tag{47b}$$

As can be seen from (26), only $\sigma_0(\tau)$ can increase with τ, whereas $\sigma_m(\tau)$ is always of the order of Γ for $m \neq 0$. Therefore we obtain

$$\phi(n,\tau) = \exp\left(-\frac{n^2}{2}\sigma_0(\tau)\right)\cdot(1 + O(n^4\Gamma^2)) \tag{48}$$

since φ_t is equally distributed on the interval 0 to 2π. Thus the characteristic function is fully determined up to first order by the averaged variance $\sigma_0(\tau)$:

$$\phi(n,\tau) = \exp\left(-\frac{n^2}{2}\sigma_0(\tau)\right). \tag{49}$$

The averaged variance can easily be calculated with (24), (26), and (43b) by the Ito calculus. Thus we obtain in first order of Γ

$$\sigma(t,\tau) = \omega_0^2\left[\int_t^{t+\tau} \alpha(y)\Gamma\alpha(y)^T dy + 2\int_t^{t+\tau} \beta(y) \cdot \int_t^y \Psi(y,z)\{\boldsymbol{R}(z)\beta(z)^T + \boldsymbol{Q}(z)\Gamma\alpha(z)^T\}\, dz\, dy\right]. \tag{50}$$

We have replaced φ with t and have introduced the amplitude fluctuation matrix

$$\boldsymbol{R}(z) = \int_{-\infty}^{z} \Psi(z,s)\boldsymbol{Q}(s)\Gamma\boldsymbol{Q}(s)^T\Psi(z,s)^T ds. \tag{51}$$

The quantity $\sigma(t,\tau)$ is periodic in the variable t with period T^0. To obtain the averaged variance $\sigma_0(\tau)$ we have to average over one period in t according to (47b). Therefore we obtain

$$\sigma_0(\tau) = \sigma_0^{(1)}(\tau) + \sigma_0^{(2)}(\tau) \tag{52a}$$

with

$$\sigma_0^{(1)}(\tau) = \frac{1}{T^0}\int_0^{T^0} \omega_0^2 \int_t^{t+\tau} \alpha(y)\Gamma\alpha(y)^T dy\, dt \tag{52b}$$

$$\sigma_0^{(2)}(\tau) = \frac{1}{T^0}\int_0^{T^0} 2\omega_0^2 \int_t^{t+\tau} \beta(y) \int_t^y \Psi(y,z) \times \{\boldsymbol{R}(z)\beta(z)^T + \boldsymbol{Q}(z)\Gamma\alpha(z)^T\}\, dz\, dy\, dt. \tag{52c}$$

Since the integrand in (52b) is also periodic we obtain, for the first part of the phase fluctuations,

$$\sigma_0^{(1)}(\tau) = d_1|\tau| \tag{53a}$$

where

$$d_1 = \omega_0^2 \frac{1}{T^0}\int_0^{T^0} \alpha(y)\Gamma\alpha(y)^T dy. \tag{53b}$$

To gain greater insight into the second part of the phase fluctuations we use the presentation of the fundamental matrix $\Psi(y,z)$ given in the Appendix:

$$\Psi(y,z) = \sum_{\iota=2}^{N} e^{\eta_\iota(y-z)}\vec{u}_\iota(y)\vec{v}_\iota^T(z) \tag{A10}$$

where η_ι is the Floquet exponent belonging to $\vec{u}_\iota(t)$, which is the periodic part of one of the fundamental solutions of the linear and time periodic differential equation (20), and $\vec{v}_\iota(t)$ is the corresponding periodic part of the fundamental solution of the adjoint equation to (20) (see the Appendix). Thus we can calculate the elements $R_{ij}^N(z)$ of the matrix $\boldsymbol{R}(z)$ on the basis of the $\vec{u}_i(z)$:

$$\boldsymbol{R}(z)=\sum_{i=2}^{N}\sum_{j=2}^{N} R_{ij}^N(z)\vec{u}_i(z)\vec{u}_j^T(z) \tag{54}$$

by inserting the representation of the fundamental matrix given in (A10) into (51). From this we obtain

$$R_{ij}^N(z)=\int_{-\infty}^{z} e^{(\eta_i+\eta_j)(z-s)}\vec{v}_i^T(s)\boldsymbol{Q}(s)\Gamma\boldsymbol{Q}(s)^T\vec{v}_j(s)\,ds. \tag{55}$$

Dividing the interval of integration in intervals of length T^0 and using the periodicity of the second part of the integrand, we obtain

$$R_{ij}^N(z)=\frac{1}{1-e^{(\eta_i+\eta_j)T^0}}\int_0^{T^0} e^{(\eta_i+\eta_j)(T^0-t)}\vec{v}_i^T(t+z)\boldsymbol{Q}(t+z) \times\Gamma\boldsymbol{Q}^T(t+z)\vec{v}_j(t+z)\,dt. \tag{56}$$

Thus we can calculate these matrix elements by integrating only over a single period. Inserting (A10) and (54) into (52c) and using the orthogonality relations (A12) in the Appendix, we obtain

$$\sigma^{(2)}(t,\tau)=2\omega_0^2\sum_{k=2}^{N}\int_t^{t+\tau}C_k(y)\int_t^y e^{\eta_k(y-z)}D_k(z)\,dz\,dy \tag{57}$$

with the periodic scalar functions

$$C_k(y)=\beta(y)\vec{u}_k(y) \tag{58a}$$

$$D_k(z)=\sum_{j=2}^{N}\left[R_{kj}^N(z)C_j(z)\right]+\vec{v}_k^T(z)\boldsymbol{Q}(z)\Gamma\alpha(z)^T. \tag{58b}$$

From (57) one can see that every fundamental solution $k\geqslant 2$ of the linear and time periodic differential equation (20) makes a contribution

$$\sigma_k^{(2)}(t,\tau)=2\omega_0^2\int_t^{t+\tau}C_k(y)\int_t^y e^{\eta_k(y-z)}D_k(z)\,dz\,dy \tag{59}$$

to the second part of the phase fluctuations:

$$\sigma^{(2)}(t,\tau)=\sum_{k=2}^{N}\sigma_k^{(2)}(t,\tau). \tag{60}$$

Thus we obtain for the contribution of the kth solution to the averaged phase fluctuations

$$\sigma_{0_k}(\tau)=\frac{1}{T^0}\int_0^{T^0}\sigma_k^{(2)}(t,\tau)\,dt. \tag{61}$$

The integrations in (59) and (61) can be carried out by substituting the coresponding Fourier series for the periodic functions $C_k(t)$ and $D_k(t)$

$$C_k(t)=\sum_{n=-\infty}^{+\infty}\hat{C}_{k,n}e^{jn\omega_0 t} \tag{62a}$$

$$D_k(t)=\sum_{n=-\infty}^{+\infty}\hat{D}_{k,n}e^{jn\omega_0 t} \tag{62b}$$

into (59). With relation (35) we obtain

$$\sigma_{0_k}(\tau)=\sum_{k=2}^{N}\sum_{n=-\infty}^{+\infty}a_{k,n}S_{k,n}(\tau) \tag{63}$$

with coefficients

$$a_{k,n}=2\omega_0^2\hat{C}_{k,n}\hat{D}_{k,-n} \tag{64}$$

and structural functions

$$S_{k,n}(\tau)=-(\eta_k+jn\omega_0)^{-2}-(\eta_k+jn\omega_0)^{-1}\tau +(\eta_k+jn\omega_0)^{-2}e^{(\eta_k+jn\omega_0)\tau}. \tag{65}$$

Now we can collect all terms and obtain the following relationship for the time dependence of the averaged phase fluctuations $\sigma_0(\tau)$:

$$\sigma_0(\tau)=A+D_\varphi|\tau|+G(|\tau|). \tag{66a}$$

Where from (52a) to (65) for the damping factor A we obtain

$$A=-\sum_{n=-\infty}^{+\infty}\sum_{k=2}^{N}a_{k,n}(\eta_k+jn\omega_0)^{-2} \tag{66b}$$

for the diffusion constant of the phase D_φ,

$$D_\varphi=\sum_{k=1}^{N}d_k \tag{66c}$$

with d_1 according to (53b) and d_k for $k>1$ by (63) and (65):

$$d_k=-\sum_{n=-\infty}^{+\infty}a_{k,n}(\eta_k+jn\omega_0)^{-1}. \tag{66d}$$

Finally for the function $G(\tau)$ we obtain

$$G(\tau)=\sum_{k=2}^{N}g_k(\tau) \tag{66e}$$

with

$$g_k(\tau)=\sum_{n=-\infty}^{+\infty}a_{k,n}(\eta_k+jn\omega_0)^{-2}e^{(\eta_k+jn\omega_0)\tau}. \tag{66f}$$

With (66a) we can calculate the spectral shape of the nth harmonic $F_n(f)$ produced by phase noise according to (42), (49), and (66a):

$$F_n(f)=\mathscr{F}\left\{\exp\left[-\frac{n^2}{2}\left(A+D_\varphi|\tau|+G(|\tau|)\right)\right]\right\}. \tag{67}$$

And we obtain with the folding theorem of the Fourier transformation

$$F_n(f)=e^{-n^2A/2}\mathscr{F}\{e^{-(n^2D_\varphi/2)|\tau|}\}*\mathscr{F}\{e^{-n^2G(|\tau|)/2}\}. \tag{68}$$

The function $G(|\tau|)$ vanishes for $\tau\to\infty$ and is always of the order of Γ for $|\tau|>0$, which means $G(|\tau|)\ll 1$ for all τ. Therefore we can expand the exponential in $G(|\tau|)$ up to first order. If we additionally assume that $n^2D_\varphi/2\ll -\mathrm{Re}\{\eta_k\}$, $2\leqslant k\leqslant N$, we obtain

$$F_n(f)=e^{-n^2A/2}\left[\mathscr{F}\{e^{-n^2D_\varphi|\tau|/2}\}-\mathscr{F}\{n^2G(|\tau|)/2\}\right]. \tag{69}$$

These assumptions are well fulfilled in electrical oscillators due to the weakness of the noise sources. The Fourier transforms in (69) can easily be carried out since we have only to transform exponentials which lead to Lorenzian lines. Therefore we obtain

$$F_n(f) = F_n^{(0)}(f) + F_n^{(1)}(f) \tag{70a}$$

$$\begin{aligned} F_n^{(0)}(f) = e^{-n^2A/2}\Bigg\{ & \frac{n^2 D_\varphi}{\left(n^2 D_\varphi/2\right)^2 + (2\pi f)^2} \\ & + n^2 \sum_{i_r} a_{i,0}\eta_i^{-2} \frac{-\eta_i}{\eta_i^2 + (2\pi f)^2} \\ & + n^2 \sum_{i_c} \mathrm{Re}\left\{ a_{i,0}\eta_i^{-2} \right\} \\ & \cdot \left[\frac{-\eta_i'}{\eta_i'^2 + (2\pi f + \eta_i'')^2} + \frac{-\eta_i'}{\eta_i'^2 + (2\pi f - \eta_i'')^2} \right] \\ & + n^2 \sum_{i_c} \mathrm{Im}\left\{ a_{i,0}\eta_i^{-2} \right\} \\ & \cdot \left[\frac{\eta_i'' + 2\pi f}{\eta_i'^2 + (2\pi f + \eta_i'')^2} + \frac{\eta_i'' - 2\pi f}{\eta_i'^2 + (2\pi f - \eta_i'')^2} \right] \Bigg\} \end{aligned} \tag{70b}$$

$$\begin{aligned} F_n^{(1)}(f) = e^{-n^2A/2}\Bigg\{ & n^2 \sum_{i_r} \sum_{m=1}^{\infty} \Bigg\{ \mathrm{Re}\left\{ a_{i,m}(\eta_i + jm\omega_0)^{-2} \right\} \\ & \times \left[\frac{-\eta_i}{\eta_i^2 + (2\pi f + m\omega_0)^2} + \frac{-\eta_i}{\eta_i^2 + (2\pi f - m\omega_0)^2} \right] \\ & - \mathrm{Im}\left\{ a_{i,m}(\eta_i + jm\omega_0)^{-2} \right\} \Bigg[\frac{m\omega_0 + 2\pi f}{\eta_i^2 + (2\pi f + m\omega_0)^2} \\ & + \frac{m\omega_0 - 2\pi f}{\eta_i^2 + (2\pi f - m\omega_0)^2} \Bigg] \Bigg\} \\ & + n^2 \sum_{i_c} \sum_{m=-\infty}^{\infty}{}' \Bigg\{ \mathrm{Re}\left\{ a_{i,m}(\eta_i + jm\omega_0)^{-2} \right\} \\ & \times \Bigg[\frac{-\eta_i'}{\eta_i'^2 + (2\pi f + \eta_i'' + m\omega_0)^2} \\ & + \frac{-\eta_i'}{\eta_i'^2 + (2\pi f - \eta_i'' - m\omega_0)^2} \Bigg] \\ & - \mathrm{Im}\left\{ a_{i,m}(\eta_i + jm\omega_0)^{-2} \right\} \\ & \cdot \Bigg[\frac{\eta_i'' + m\omega_0 + 2\pi f}{\eta_i'^2 + (2\pi f + \eta_i'' + m\omega_0)^2} \\ & + \frac{\eta_i'' + m\omega_0 - 2\pi f}{\eta_i'^2 + (2\pi f - \eta_i'' - m\omega_0)^2} \Bigg] \Bigg\} \Bigg\}. \end{aligned} \tag{70c}$$

The summation over i with indices r and c, respectively, denotes summation over those i's with real Floquet exponents η_i and a pair of conjugate complex Floquet exponents $\eta_i = \eta_i' \pm j\eta_i''$. Also we have used the relations $a_{i,m}^* = a_{i+1,-m}$, where η_i and η_{i+1} are a pair of conjugate complex Floquet exponents. As we see from (70b) the main part of the spectrum is a Lorenzian line with 3 dB half-bandwidth $\Delta f_{3\,\mathrm{dB}} = n^2 D_\varphi / 4\pi$. The other parts of $F_n^{(0)}(f)$ are also centered on $f = 0$, where those parts with $-\mathrm{Re}\{\eta_i\} \ll \omega_0$ dominate. Therefore this function is strongly peaked at $f = 0$. $F_n^{(1)}(f)$ consists of those parts centered on $f = mf_0$, $m \neq 0$. And in this function the terms with $-\mathrm{Re}\{\eta_i\} \gg \omega_0$ cause a broad spectrum. Thus $F_n^{(1)}(f)$ only contribute to the noise floor of an oscillator. This noise floor is produced by the fast relaxing normal modes. From (70a)–(70c) one can see that the phase noise spectrum is fully determined by the Floquet exponents and the coefficients $a_{k,m}$, which determine the coupling of the noise sources to the kth mode of the time periodic system and the conversion into phase fluctuations.

B. Determination of the Correlation Between Amplitude and Phase

From (30) and (31) the following identities can be derived:

$$C_{\Delta x x^0}(\tau) = C_{x^0 \Delta x}(-\tau)^T \tag{71a}$$

$$\hat{C}_{\Delta x x^0}(f) = \hat{C}_{x^0 \Delta x}(f)^\dagger. \tag{71b}$$

Thus we can restrict ourselves to calculate $C_{x^0\Delta x}(\tau)$. From (30) and (33) we obtain

$$C_{x^0\Delta x}(\tau) = \sum_{m=-\infty}^{+\infty} \vec{A}_m e^{jm\omega_0\tau} \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle e^{jm\omega_0(t+\vartheta(t))} \cdot e^{jm\omega_0(\vartheta(t+\tau)-\vartheta(t))} \Delta\vec{x}_\perp^T(t+\vartheta(t)) \rangle \, dt. \tag{72}$$

First we average over all realizations of the processes with the same $\vartheta(t)$ and later we take the average about $\vartheta(t)$. Therefore we calculate the average

$$\vec{K}^T = \langle e^{jm\omega_0(\vartheta(t+\tau)-\vartheta(t))} \Delta\vec{x}_\perp^T(t+\vartheta(t)) \rangle_{\vartheta(t)=\mathrm{const}}. \tag{73}$$

We introduce the characteristic time constants τ_H and τ_c, which are defined as follows:

$$\sigma_0(\tau_H) = (2\pi)^2 \tag{74a}$$

$$\tau_c = 1/\mathrm{Re}\{-\eta\} \tag{74b}$$

where η is the largest negative Floquet exponent of the fundamental matrix $\boldsymbol{\Psi}$. Thus τ_c is the correlation time between the amplitude fluctuations, and τ_H is the time after which the phase fluctuations cover the whole limit cycle. From (66a), we obtain for the time $\tau_H = (2\pi)^2 / D_\varphi = \pi/\Delta f_{3\,\mathrm{dB}}$ $(n = 1)$. If the intensities of the noise sources are small, the relation $\tau_c \ll \tau_H$ holds. In electrical oscillators this condition is excellently fulfilled. Therefore we can separate $\vec{K}^T$ as follows:

$$\vec{K}^T = \langle e^{jm\omega_0(\vartheta(t+\tau)-\vartheta(t+\tau_c))} \cdot \langle e^{jm\omega_0(\vartheta(t+\tau_c)-\vartheta(t))} \Delta\vec{x}_\perp^T(t+\vartheta(t)) \rangle \rangle_{\vartheta(t)=\mathrm{const}}. \tag{75}$$

since the amplitude is not correlated with the increase of phase at times later than τ_c. On the other hand $\sigma_0(\tau_c) \ll 1$. Therefore we can expand the exponential function in the inner average in (75) and we can also neglect τ_c in the first exponential function for $\tau \gg \tau_c$. In this way we obtain, with (24), (26), and (37), up to first order in Γ,

$$\vec{K}(\tau \gg \tau_c)^T = jm\omega_0 \phi(m,\tau)\vec{H}(z,\tau)^T \tag{76}$$

with

$$\vec{H}(z,\tau)^T = \begin{cases} \int_z^{z+\tau} \beta(y)\Psi(y,z)\boldsymbol{R}(z)\,dy, & \tau \geqslant 0 \\ \int_z^{z-|\tau|}\{\beta(y)\boldsymbol{R}(y) \\ \quad +\alpha(y)\Gamma \boldsymbol{Q}(y)^T\}\Psi(z,y)^T dy, & \tau < 0 \end{cases} \tag{77}$$

and $z = t+\vartheta(t)$. $\vec{H}(z,\tau)$ is periodic with period T^0 with respect to the variable z. Thus we obtain for the correlation matrix $\boldsymbol{C}_{x^0\Delta x}$, according to (72), (73), and (76),

$$\boldsymbol{C}_{x^0\Delta x}(\tau) = \sum_{m=-\infty}^{+\infty} jm\omega_0\phi(m,\tau)\vec{A}_m \vec{H}_{-m}(\tau)^T e^{jm\omega_0\tau} \tag{78}$$

where we have set

$$\vec{H}_n = \frac{1}{T^0}\int_0^{T^0} \vec{H}(z,\tau)e^{-jn\omega_0 z}\,dz. \tag{79}$$

This result is also valid for $\tau < \tau_c$, as one can easily see from (75)–(78), since $\phi(m,\tau) \cong 1$ for $\tau < \tau_c$. The multiplication of $\vec{H}_m(\tau)$ with $\phi(m,\tau)$ is necessary to avoid singularities in the spectrum, because $\vec{H}_m(\tau)$ does not vanish for $\tau \to \infty$. With the relation (A10) we obtain from (77),

$$\vec{H}(z,\tau)^T = \begin{cases} \sum_{k=2}^{N}\int_z^{z+\tau} C_k(y)\vec{r}_k^T(z)e^{\eta_k(y-z)}\,dy \\ \sum_{k=2}^{N}\int_z^{z-|\tau|} D_k(y)\vec{u}_k^T(z)e^{\eta_k(z-y)}\,dy \end{cases} \tag{80}$$

with $C_k(y)$, $D_k(y)$ according to (58a) and (58b), and

$$\vec{r}_k^T(z) = \sum_{j=2}^{N} R_{kj}^N(z)\vec{u}_j^T(z). \tag{81}$$

With the Fourier expansions for $C_k(y)$, $D_k(y)$ according to (62a) and (62b), and

$$\vec{r}_k^T(z) = \sum_{m=-\infty}^{+\infty} \hat{\vec{r}}_{k,m}^T e^{jm\omega_0 z} \tag{82a}$$

$$\vec{u}_k^T(z) = \sum_{l=-\infty}^{+\infty} \hat{\vec{u}}_{k,l}^T e^{jl\omega_0 z} \tag{82b}$$

we obtain for $\vec{H}_n(\tau)^T$ according to (79)–(82b)

$$\vec{H}_n(\tau)^T = \begin{cases} \sum_{k=2}^{N}\sum_{l=-\infty}^{+\infty} \vec{q}_{k,n,l}^{T+}(e^{(\eta_k+jl\omega_0)\tau}-1), & \tau \geqslant 0 \\ \sum_{k=2}^{N}\sum_{l=-\infty}^{+\infty} \vec{q}_{k,n,l}^{T-}(e^{(\eta_k-jl\omega_0)|\tau|}-1), & \tau < 0 \end{cases} \tag{83}$$

where

$$\vec{q}_{k,n,l}^{T+} = \hat{C}_{k,l}\hat{\vec{r}}_{k,n-l}^T(\eta_k+jl\omega_0)^{-1}$$
$$\vec{q}_{k,n,l}^{T-} = -\hat{D}_{k,l}\hat{\vec{u}}_{k,n-l}^T(\eta_k-jl\omega_0)^{-1}. \tag{84}$$

By (83) and (66a), we obtain up to first order in Γ

$$\phi(m,\tau)\vec{H}_{-m}(\tau)^T = \sum_{l=-\infty}^{+\infty}\sum_{k=2}^{N} \vec{q}_{k,-m,l}^{T\pm} \cdot (e^{(\eta_k \pm jl\omega_0)|\tau|}-1)(e^{-m^2(A+D_\varphi|\tau|)/2}) \tag{85}$$

where the plus sign holds for $\tau \geqslant 0$ and the minus sign for $\tau < 0$. By Fourier transformation we obtain, with (78),

$$\hat{\boldsymbol{C}}_{x^0\Delta x}(f) = e^{-n^2A/2}\sum_{m=-\infty}^{+\infty} jm\omega_0 \sum_{k=2}^{N}\vec{A}_m \cdot\left[\vec{S}_{k,-m}^{T+}(f)+\vec{S}_{k,-m}^{T-}(f)\right] \tag{86}$$

with

$$\vec{S}_{k,-m}^{T+}(f) = \sum_{l=-\infty}^{+\infty} q_{k,-m,l}^{T+} \cdot\left[\frac{1}{-\eta_k' + m^2D_\varphi/2 + j(2\pi f - \eta_k'' - (m+l)\omega_0)} - \frac{1}{m^2D_\varphi/2 + j(2\pi f - m\omega_0)}\right]$$

$$\vec{S}_{k,-m}^{T-}(f) = \sum_{l=-\infty}^{+\infty} q_{k,-m,l}^{T-} \cdot\left[\frac{1}{-\eta_k' + m^2D_\varphi/2 - j(2\pi f + \eta_k'' - (m+l)\omega_0)} - \frac{1}{m^2D_\varphi/2 - j(2\pi f - m\omega_0)}\right]. \tag{87}$$

With (71b) and (86) the correlation spectrum due to phase amplitude correlations is known.

C. Determination of Amplitude Noise

The last term in (28) describes the amplitude noise. From (32) follows

$$\boldsymbol{C}_{\Delta x\Delta x}(-\tau) = \boldsymbol{C}_{\Delta x\Delta x}(\tau)^T \tag{88a}$$

$$\hat{\boldsymbol{C}}_{\Delta x\Delta x}(f) = \mathscr{F}\{\theta(\tau)\boldsymbol{C}_{\Delta x\Delta x}(\tau)\} + \mathscr{F}\{\theta(\tau)\boldsymbol{C}_{\Delta x\Delta x}(\tau)\}^\dagger \tag{88b}$$

with

$$\theta(\tau) = \begin{cases} 1 & \tau \geqslant 0 \\ 0, & \tau < 0. \end{cases} \tag{88c}$$

Thus we can restrict ourselves to the case $\tau > 0$. With (32), (24), and (51) we obtain

$$\boldsymbol{C}_{\Delta x\Delta x}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle \Psi(t+\tau+\vartheta(t+\tau), t+\vartheta(t))\boldsymbol{R}(t+\vartheta(t))\rangle\,dt. \tag{89}$$

Now we replace $t+\vartheta(t)$ with z and the differential dt with dz. Since the fundamental matrix is only nonzero for $\tau<\tau_c$, we can neglect the time difference $\vartheta(t+\tau)-\vartheta(t)$ in the argument of $\boldsymbol{\Psi}$. Thus we obtain

$$C_{\Delta x \Delta x}(\tau)=\frac{1}{T^0}\int_0^{T^0}\boldsymbol{\Psi}(z+\tau,z)\boldsymbol{R}(z)\,dz \tag{90}$$

since the integrand is periodic in z with period T^0. With (A10), (54), and (81) we obtain

$$C_{\Delta x \Delta x}(\tau)=\frac{1}{T^0}\int_0^{T^0}\sum_{k=2}^{N} e^{\eta_k\tau}\vec{u}_k(z+\tau)\vec{r}_k^{\,T}(z)\,dz. \tag{91}$$

By the Fourier expansions according to (82a) and (82b) we obtain

$$C_{\Delta x \Delta x}(\tau)=\sum_{k=2}^{N}\sum_{m=-\infty}^{+\infty}\vec{u}_{k,m}\vec{r}_{k,-m}^{\,T}e^{(\eta_k\tau+jm\omega_0)\tau}. \tag{92}$$

The correlation spectrum of the amplitude noise can be calculated by (88b) and the expression for the correlation matrix (92) for time $\tau>0$

$$\hat{C}_{\Delta x \Delta x}(f)=\sum_{k=2}^{N}\sum_{m=-\infty}^{+\infty}\left[\vec{u}_{k,m}\vec{r}_{k,-m}^{\,T}\frac{1}{-\eta_k'+j(2\pi f-\eta_k''-m\omega_0)}+\vec{r}_{k,m}\vec{u}_{k,-m}^{\,T}\frac{1}{-\eta_k'-j(2\pi f+\eta_k''-m\omega_0)}\right] \tag{93}$$

where we have used the relations $\vec{u}_{k,m}^{\,*}=\vec{u}_{k+1,-m}$ and $\vec{r}_{k,-m}^{\,*}=\vec{r}_{k+1,m}$. Now the complete spectrum of an oscillator is formally derived and we can tackle the problem of the numerical determination of the correlation spectra given above.

V. Numerical Procedure for the Determination of the Correlation Spectrum

In this section a numerical procedure for the determination of the correlation matrices calculated in Section IV is given. Therefore we replace the continuous stochastic processes derived in Section III by time discrete processes. The discretization in time is achieved by Poincaré mapping [25], as shown in Fig. 2. We assume that the limit cycle $\vec{x}^{\,0}(t)$ with period T^0 is known. The limit cycle is a curve in $\mathbb{R}^N$, which is parameterized by time from $0\leqslant t<T^0$. We choose M equidistant points $\vec{x}_i^{\,0}$ on the limit cycle, as follows:

$$\vec{x}_i^{\,0}=\vec{x}^{\,0}(t_i)\quad\text{with}\quad t_i=i\cdot\Delta t,\ 1\leqslant i\leqslant M,\ \Delta t=\frac{T^0}{M}. \tag{94}$$

By these points the hyperplanes $\mathscr{N}_i$ with $\vec{x}_i^{\,0}\in\mathscr{N}_i$ and normal vectors

$$\vec{n}_i=\frac{\dot{\vec{x}}_i^{\,0}}{\|\dot{\vec{x}}_i^{\,0}\|},\qquad\text{where } \dot{\vec{x}}_i^{\,0}=\vec{F}\left(\vec{x}_i^{\,0}\right) \tag{95}$$

are defined. The flux of the unperturbed differential equation (2) describes a nonlinear mapping of a deviation $\Delta\vec{x}_{i-1}^{\,0}$ at point $\vec{x}_{i-1}$ onto a deviation $\Delta\vec{x}_i$ at point $\vec{x}_i^{\,0}$

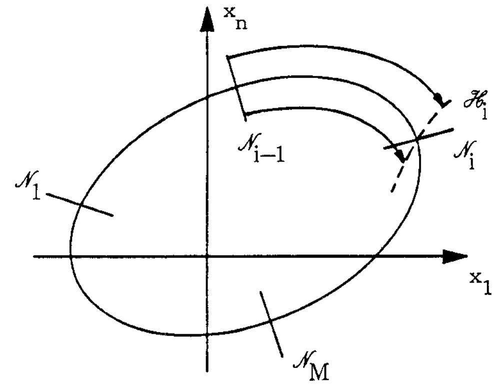

Fig. 2. Time discretization by Poincaré mapping.

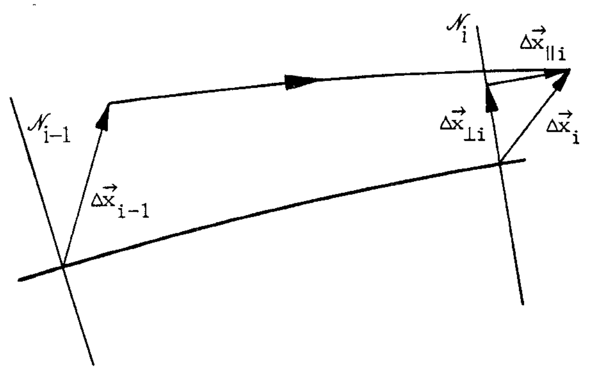

Fig. 3. Mapping of deviation by the flux of the unperturbed system.

during a time interval Δt. That is,

$$\Delta\vec{x}_i=\vec{B}_i\left(\Delta\vec{x}_{i-1}\right). \tag{96}$$

See Fig. 3. In particular the hyperplane $\mathscr{N}_{i-1}$ is mapped onto the curved area $\mathscr{H}_i$, (see Fig. 2). For small deviations $\Delta\vec{x}_{i-1}$ the nonlinear map $\vec{B}_i$ can be linearized. Thus we obtain the matrix $\boldsymbol{B}_i$ with elements

$$(\boldsymbol{B}_i)_{kl}=\frac{\partial(\Delta\vec{x}_i)_k}{\partial(\Delta\vec{x}_{i-1})_l} \tag{97}$$

and with (96)

$$\Delta\vec{x}_i=\boldsymbol{B}_i\Delta\vec{x}_{i-1}. \tag{98}$$

Now we separate the map which describes the evolution of the normal deviation from the unperturbed orbit. Therefore we introduce the projection operator $\boldsymbol{P}_i$ by

$$\boldsymbol{P}_i=\left(\boldsymbol{1}-\vec{n}_i\vec{n}_i^{\,T}\right) \tag{99}$$

and obtain for the normal deviation

$$\Delta\vec{x}_{\perp i}=\boldsymbol{A}_i\Delta\vec{x}_{\perp i-1} \tag{100a}$$

where

$$\boldsymbol{A}=\boldsymbol{P}_i\boldsymbol{B}_i. \tag{100b}$$

By multiplication of the mapping matrices $\boldsymbol{A}_i$ over one period,

$$\boldsymbol{A}_g=\prod_{i=1}^{M}\boldsymbol{A}_i \tag{101}$$

we obtain the matrix $\boldsymbol{A}_g$, which describes the time evolution of a normal deviation $\Delta\vec{x}_\perp$ after one revolution around the limit cycle (see Fig. 4).

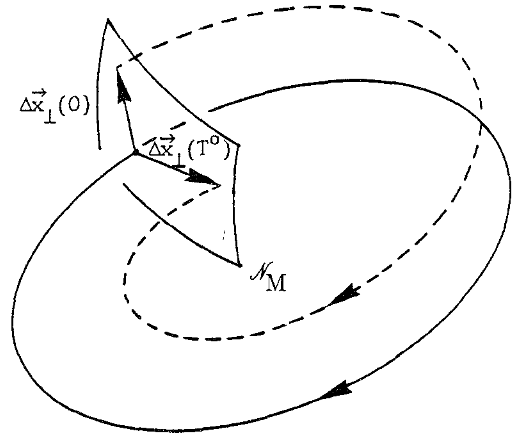

Fig. 4. Mapping of a deviation after one revolution around the limit cycle $\vec{x}^0$ by the Poincaré map A_g.

This linearized Poincaré map A_g is equivalent to the fundamental matrix

$$\Psi(T^0,0) = A_g \tag{102}$$

which also describes the time evolution of the normal deviation over one period. With (A10) of the Appendix we obtain

$$A_g = \sum_{i=2}^{N} e^{\eta_i T^0} \vec{u}_i(0)\vec{v}_i^T(0) \tag{103}$$

where we have used $\vec{u}_i(T^0) = \vec{u}_i(0)$ due to the periodicity of the $\vec{u}_i$.

From this relation and with (A12) we can see that $\vec{u}_i(0)$ and $\vec{v}_i(0)$ are the right- and left-sided eigenvectors of the matrix A_g:

$$A_g \vec{u}_i(0) = \lambda_i \vec{u}_i(0) \tag{104a}$$

$$\vec{v}_i^T(0) A_g = \lambda_i \vec{v}_i^T(0) \tag{104b}$$

with eigenvalue $\lambda_i = e^{\eta_i T^0}$.

The solutions $\vec{u}_i(t)$ and $\vec{v}_i(t)$ can be determined subsequently at the points $t = j\Delta t$ by the recurrence relations

$$\vec{u}_i(j\Delta t) = e^{-\eta_i \Delta t} A_j \vec{u}_i((j-1)\Delta t) \tag{105a}$$

$$\vec{v}_i^T(j\Delta t) = e^{-\eta_i \Delta t} \vec{v}_i^T((j+1)\Delta t) A_{j+1} \tag{105b}$$

since

$$A_i = \Psi(i\Delta t, (i-1)\Delta t). \tag{106}$$

The vectors $\vec{v}_i^T(0)$ have to be normalized at the beginning of the iteration to fulfill relation (A12). The matrices β_i can be obtained by numerical differentiation techniques according to (23a). Therefore with this construction of the basis $\vec{u}_i$ and the corresponding basis $\vec{v}_i^T$ of the dual space, we can calculate the spectra derived above. The integrations which have to be carried out above can be done by interpolation of the integrand with cubic splines.

VI. Numerical Example: Colpitts Oscillator

To convey an idea of the applicability of the theory described above we will calculate the phase noise spectrum of the Colpitts oscillator shown in Fig. 5 in the neighborhood of the oscillation frequency f_0, which is given by the function $F_1^{(0)}(f - f_0)$. The transistor BFR 35 A is modeled by a Ebers–Moll equivalent circuit [26], shown in Fig. 6. Transistor parameters are taken from the paper by Schwaderer [27]. From the lumped circuit models shown in Figs. 5 and 6, we can derive the equations of motion for the state vector $\vec{x} = (i_1, i_2, u_1, u_2, u_3, u_4) \in \mathbb{R}^6$:

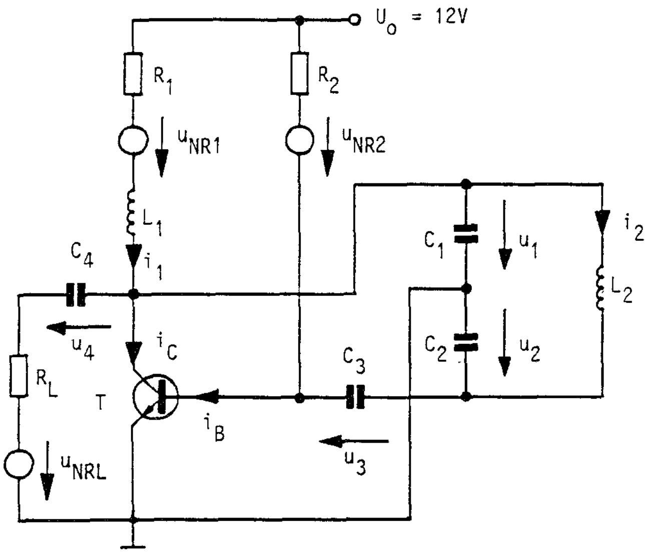

Fig. 5. Analyzed Colpitts oscillator circuit. The values of the components are $R_1 = 350$ Ω, $R_2 = 110$ kΩ, $R_L = 500$ Ω, $L_1 = 10$ μH, $L_2 = 30$ nH, $C_1 = 10$ pF, $C_2 = 940$ pF, $C_3 = 2.7$ nF, $C_4 = 1.5$ nF, $T =$ BFR 35 A.

$$\frac{d}{dt} i_1 = -\frac{1}{L_1}(R_1 i_1 + u_1 - u_0) + G_1(\vec{x})\vec{\xi} \tag{107}$$

$$\frac{d}{dt} i_2 = \frac{1}{L_2}(u_1 + u_2) + G_2(\vec{x})\vec{\xi} \tag{108}$$

$$\frac{d}{dt} u_1 = \frac{1}{C_1}\left(i_1 - i_2 - i_c + \frac{1}{R_L}(u_4 - u_1)\right) + G_3(\vec{x})\vec{\xi} \tag{109}$$

$$\frac{d}{dt} u_2 = \frac{1}{C_2}\left(i_b - i_2 - \frac{1}{R_2}(u_0 + u_2 + u_3)\right) + G_4(\vec{x})\vec{\xi} \tag{110}$$

$$\frac{d}{dt} u_3 = \frac{1}{C_3}\left(i_b - \frac{1}{R_2}(u_0 + u_2 + u_3)\right) + G_5(\vec{x})\vec{\xi} \tag{111}$$

$$\frac{d}{dt} u_4 = \frac{1}{R_L C_4}(u_1 - u_4) + G_6(\vec{x})\vec{\xi} \tag{112}$$

where the currents in the transistor are given by

$$i_c = i_{CE} - i_{BC} \tag{113}$$

$$i_b = i_{BE} + i_{BC} \tag{114}$$

$$i_{CE} = \frac{I_s}{Q_b/Q_{b0}}\left[\exp(u_{BE}/u_T) - \exp(u_{BC}/u_T)\right] \tag{115}$$

$$i_{BE} = \frac{I_s}{\beta_N}\left[\exp(u_{BE}/u_T) - 1\right] \tag{116}$$

$$i_{BC} = \frac{I_s}{\beta_I}\left[\exp(u_{BC}/u_T) - 1\right] \tag{117}$$

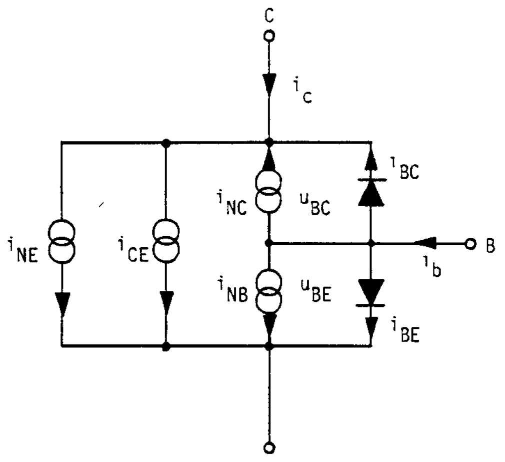

Fig. 6. Ebers–Moll equivalent circuit for the transistor BFR 35 A. $I_s = 1.27 \cdot 10^{-16}$ A, $\beta_I = 5.5$, $\beta_N = 140$, $U_A = 15$ V, $U_B = 4.3$ V.

with

$$u_{BE} = -u_2 - u_3 \tag{118}$$

$$u_{BC} = -u_1 - u_2 - u_3 \tag{119}$$

$$\frac{Q_b}{Q_{b0}} = 1 + \frac{U_{BE}}{U_b} + \frac{U_{BC}}{U_a}. \tag{120}$$

The thermal noise sources of the resistors and the shot noise sources of the pn junctions in the transistor produce the stochastic forces $G_i(\vec{x})\vec{\xi}$, which are given by

$$G_1(\vec{x})\vec{\xi} = -\frac{1}{L_1} u_{NRL} \tag{121}$$

$$G_2(\vec{x})\vec{\xi} = 0 \tag{122}$$

$$G_3(\vec{x})\vec{\xi} = \frac{1}{C_1 R_L} u_{NRL} - \frac{1}{C_1} i_{NE} + \frac{1}{C_1} i_{NC} \tag{123}$$

$$G_4(\vec{x})\vec{\xi} = \frac{1}{C_2 R_2} u_{NR2} + \frac{1}{C_2} i_{NB} + \frac{1}{C_2} i_{NC} \tag{124}$$

$$G_5(\vec{x})\vec{\xi} = \frac{1}{C_3 R_2} u_{NR2} + \frac{1}{C_3} i_{NB} + \frac{1}{C_3} i_{NC} \tag{125}$$

$$G_6(\vec{x})\vec{\xi} = -\frac{1}{C_4 R_L} u_{NRL}. \tag{126}$$

For the thermal and shot noise sources we obtain

$$u_{NR1} = \sqrt{2kTR_1}\,\xi_1 \tag{127}$$

$$u_{NR2} = \sqrt{2kTR_2}\,\xi_2 \tag{128}$$

$$u_{NRL} = \sqrt{2kTR_L}\,\xi_3 \tag{129}$$

$$i_{NE} = \sqrt{q|i_{CE}|}\,\xi_4 \tag{130}$$

$$i_{NB} = \sqrt{q|i_{BE}|}\,\xi_5 \tag{131}$$

$$i_{NC} = \sqrt{q|i_{BC}|}\,\xi_6. \tag{132}$$

From (126) to (132) we obtain the matrix elements

$$G_{ij}(\vec{x}) = \frac{\partial\left(G_i\vec{\xi}\right)}{\partial \xi_j} \tag{133}$$

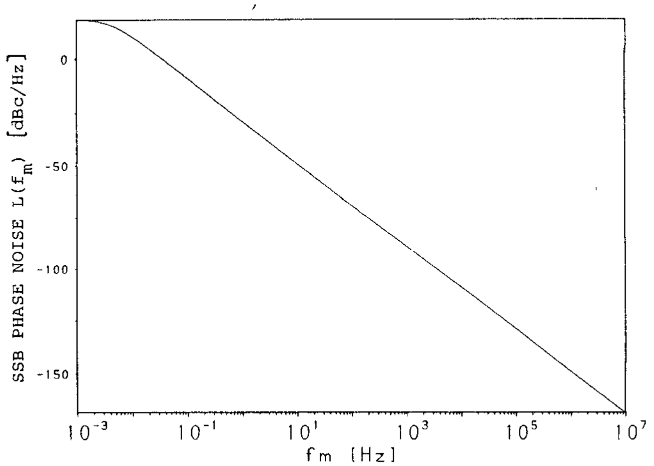

Fig. 7. Single sideband phase noise of the Colpitts oscillator shown in Fig. 5.

where the ξ_i are normalized white Gaussian noise sources with correlation coefficients

$$\langle \xi_i \xi_j \rangle = \delta_{i,j}. \tag{134}$$

Thus the system equations are of the type discussed above. Note that those matrix elements $G_{ij}(\vec{x})$ which describe the influence of the shot noise sources in the transistor explicitly depend on the state of the system via (115)–(117) and (130)–(133) as discussed in Section II.

Application of the theory presented above results in the phase noise spectrum $F_1^{(0)}(f)$. The power spectrum of the oscillator circuit shown in Fig. 5 is proportional to the correlation spectrum of the current through the load resistor R_L. For the interesting frequency range around the center frequency of oscillation $f_0 = 292$ MHz for the oscillator shown above, we can neglect the coupling capacitor C_4. Therefore the power spectrum is directly proportional to the correlation spectrum of the state variable u_1, the collector–emitter voltage at the transistor. Thus the single sideband phase noise to carrier ratio $L(f_m)$ is given by

$$L(f_m) = F_1^{(0)}(f_m) \tag{135}$$

where f_m is the deviation from the center frequency f_0.

This single sideband phase noise to carrier ratio for the Colpitts oscillator shown above is shown in Fig. 7. As one can see the spectrum consists of a Lorenzian line with the 3 dB bandwidth $\Delta f_{3\text{dB}} = 0.4 \cdot 10^{-2}$ Hz. The main portion of this line width is caused by the conversion of amplitude fluctuations into phase fluctuations. The additional terms produced by the function $G(|\tau|)$ are completely suppressed by this Lorenzian line. Thus in the vicinity of the carrier the spectrum is completely defined by the 3 dB bandwidth. Therefore it is interesting to investigate the influence of the various noise sources on the 3 dB bandwidth. In Table I the contribution of every noise source is shown. The greatest contribution, about 90 percent, is made by the shot noise of the collector–emitter current i_{CE}. Also the shot noise of the reverse basis-collector current i_{BC} makes a contribution of about 6.2 percent to the bandwidth. This is caused by the lack of a countercoupling resistor in the

TABLE I
CONTRIBUTION OF THE VARIOUS NOISE SOURCES TO THE 3 dB BANDWIDTH $\Delta f_{3\,\mathrm{dB}}$

noise source	contribution to Δf_{3dB}	relative contribution
u_{nr1}	$0.15\cdot10^{-7}$ Hz	$3.6\cdot10^{-4}$%
u_{nr2}	$0.43\cdot10^{-6}$ Hz	10^{-2} %
u_{nr1}	$0.59\cdot10^{-4}$ Hz	1.5%
i_{ne}	$0.36\cdot10^{-2}$ Hz	90%
inb	$0.09\cdot10^{-3}$ Hz	2.2%
inc	$0.25\cdot10^{-3}$ Hz	6.2%

$\Delta f_{3\,\mathrm{dB}} = 0.4\times10^{-2}$ Hz.

emitter terminal, which allows the transistor to operate for a short time in the reverse direction. The thermal noise of the resistors makes a contribution of only about 1.5 percent to the bandwidth. From this example we can see that the method described above is appropriate for simulating the noise behavior of oscillator circuits, as is possible for linear networks today.

VII. CONCLUSIONS

The main result of this paper is the derivation of the complete correlation spectrum of an oscillator in the low-noise case. The methods used analytically for the derivation are well suited for numerical calculation of noise spectra of technically relevant oscillators, as is shown by the numerical example above. Thus with the derived procedure the complete power spectrum of an oscillator can be calculated from the lumped circuit model with the inherent noise sources of the active and passive components. The main effort of computation, the determination of the limit cycle and the mapping matrices, can be reused for further calculations with modified intensities of the noise sources. The technically important case of the influence of $1/f^{\alpha}$ noise sources on the correlation spectrum of an oscillator has not been covered in a tractable way so far. This is because the production of $1/f^{\alpha}$ noise out of white sources as is discussed in Section II for colored noise sources would lead to a linear system of infinite dimension [28], [29]. But $1/f^{\alpha}$ noise sources also can be simulated within this framework with almost no additional computational effort, as will be shown in a subsequent paper.

APPENDIX

In this appendix we will prove the representation of the fundamental matrix Ψ used before. We assume a nonlinear dynamical system

$$\dot{\vec{x}} = F(\vec{x}), \qquad \vec{x}\in\mathbb{R}^N \tag{A1}$$

with a stable limit cycle $\vec{x}^0(t)$ of period T^0. Therefore the time evolution of small deviations $\Delta\vec{x}$ from the limit cycle is given by the time periodic linear differential equation

$$\Delta\dot{\vec{x}}(t) = \mathbf{DF}(\vec{x}^0(t))\Delta\vec{x}(t) \tag{A2}$$

with $\mathbf{DF}(\vec{x}^0)$ according to (14).

From the theory of linear differential equations with periodic coefficients it is well known that there exists a set of N linear independent solutions $\vec{q}_i(t)$ to (A2) [30] with

$$\vec{q}_i(t) = e^{\eta_i t}\vec{w}_i(t) \tag{A3}$$

where η_i are the Floquet exponents and $\vec{w}_i(t)$ are periodic vectors with period T^0. The adjoint equation to (A2),

$$\Delta\dot{\vec{x}}(t)^T = -\Delta\vec{x}(t)^T\mathbf{DF}(\vec{x}^0(t)) \tag{A4}$$

has the set of solutions $\vec{p}_i^T$ where

$$\vec{p}_i(t)^T = e^{-\eta_i t}\vec{v}_i^T(t) \tag{A5}$$

with the same Floquet exponents η_i as above and the $\vec{v}_i^T(t)$ and $\vec{w}_i(t)$ fulfill the orthogonality relations

$$\vec{v}_i^T(t)\vec{w}_j(t) = \delta_{i,j} \tag{A6}$$

if the initial conditions $\vec{v}_i(0)$ are chosen according to (A6). Here we will not consider the case of a non-semisimple fundamental matrix $\Psi(T^0,0)$, which leads to multiple Floquet exponents, since semisimplicity is a generic property of the matrices in $\mathbb{R}^N\times\mathbb{R}^N$ [31]. That means that almost all matrices are diagonalizable.

From (A3) and (A6) one can easily see that the fundamental matrix $\Xi(t,s)$ of (A2) is given by

$$\Xi(t,s) = \sum_{i=1}^{N} e^{\eta_i(t-s)}\vec{w}_i(t)\vec{v}_i^T(s). \tag{A7}$$

Thus $\Xi(t,s)$ maps the initial value $\Delta\vec{x}(s)$ onto the solution $\Delta\vec{x}(t)$ of (A2):

$$\Delta\vec{x}(t) = \Xi(t,s)\Delta\vec{x}(s). \tag{A8}$$

As can be seen from (A1) and (A2) by differentiation of $\vec{x}^0(t)$, $\dot{\vec{x}}^0(t)$ can be taken as $\vec{w}_1(t)$ with corresponding Floquet exponent $\eta_1 = 0$. Therefore from (A6) we can claim that the set $\vec{v}_2(t)$ to $\vec{v}_N(t)$ completely spans the orthogonal complement $\mathscr{N}(t)$ to the tangent space at the limit cycle $\vec{x}^0(t)$.

Equation (20) describes the time evolution of the normal deviation $\Delta\vec{x}_\perp(t)$, which is given by application of the projection operator $\boldsymbol{P}(t)$ according to (17):

$$\Delta\vec{x}_\perp(t) = \boldsymbol{P}(t)\Delta\vec{x}(t). \tag{A9}$$

Therefore the fundamental matrix $\Psi(t,s)$ of (20) is given by

$$\Psi(t,s) = \boldsymbol{P}(t)\Xi(t,s) = \sum_{i=2}^{N} e^{\eta_i(t-s)}\vec{u}_i(t)\vec{v}_i^T(s) \tag{A10}$$

with $\vec{u}_i(t) = \boldsymbol{P}(t)\vec{w}_i(t)$ and $\boldsymbol{P}(t)\vec{w}_1(t) = \vec{0}$. By differentiation of $\Psi(t,s)$ with respect to t one can show that the relation

$$\dot{\Psi}(t,s) = \boldsymbol{V}(t)\Psi(t,s) \tag{A11}$$

is fulfilled. Also the orthogonality relations between the vectors $\vec{u}_i$ and $\vec{v}_i$ are saved:

$$\vec{v}_i(t)^T\vec{u}_j(t) = [\boldsymbol{P}(t)\vec{v}_i(t)]^T\vec{w}_j(t) = \vec{v}_i(t)^T\vec{w}_j(t) = \delta_{i,j} \tag{A12}$$

for $2 \leqslant i,\ j \leqslant N$.

Acknowledgment

The author wishes to thank Prof. Dr. P. Russer for many helpful discussions.

References

[1] R. L. Stratonovich, *Introduction to the Theory of Random Noise*. New York, London: Gordon and Breach, 1963.

[2] M. Lax, "Classical noise—V: Noise in self-sustained oscillations," *Phys. Rev.*, vol. 160, pp. 290–307, 1967.

[3] W. A. Edson, "Noise in oscillators," *Proc. IRE*, vol. 48, pp. 1454–1466, 1960.

[4] K. Kurokawa, "Some basic characteristics of broadband negative resistance oscillator circuits," *Bell Syst. Tech. J.*, vol. 48, pp. 1937–1955, 1969.

[5] K. Kurokawa, "Noise in synchronized oscillators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 234–240, 1968.

[6] E. Hafner, "The effects of noise in oscillators," *Proc. IEEE*, vol. 54, pp. 1311–1330, 1964.

[7] J. A. Mullen, "Background noise in nonlinear oscillators," *Proc. IEEE*, vol. 48, pp. 1467–1473, 1960.

[8] H. Ihringer, "Rauschen von Oszillatoren mit verzögerter Amplitudenregelung," dissertation, Technische Universität München, 1981.

[9] U. Penning, "Rauschspektren von einkreisigen Oszillatoren mit beliebiger Trägheit des aktiven Elementes," dissertation, Technische Universität München, 1983.

[10] K. März, "Phasen-und Amplitudenschwankungen in Oszillatoren," *Arch. Elek. Übertragung.*, vol. 24, pp. 477–490, 1970.

[11] S. Smale, "Foundations of electrical circuit theory," *J. Differential Geometry*, vol. 7, pp. 193–210, 1972.

[12] B. Callen and T. A. Welton, "Irreversibility and generalized noise," *Phys. Rev.*, vol. 83, pp. 34–40, 1951.

[13] W. B. Davenport and W. L. Roots, *Introduction to the Theory of Random Signals and Noise*. New York: McGraw-Hill, 1958, pp. 145–170.

[14] L. Arnold, *Stochastische Differentialgleichungen, Theorie und Anwendung*. München, Wien: R. Oldenburg Verlag, 1973, pp. 93–108.

[15] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Science*. Berlin: Springer Verlag, 1985, pp. 80–115.

[16] H. Haken, "Nonequilibrium phase transitions and bifurcation of limit cycles and multiperiodic flows," *Z. Phys.*, vol. B 29, pp. 61–66, 1978.

[17] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*. Oxford: Oxford University Press, 1977, pp. 231–237.

[18] G. E. Uhlenbeck and L. S. Ornstein, "On the theory of the Brownian motion," *Phys. Rev.*, vol. 36, pp. 830–841, 1930.

[19] A. Einstein, "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen," *Ann. Phys.* (Leipzig), vol. 17, pp. 549–560, 1905.

[20] A. Khintchine, "Korrelationstheorie der stat. stochastischen Prozesse," *Math. Ann.*, vol. 109, pp. 604–615, 1934.

[21] L. Mandel, "Interpretation of instantaneous frequencies," *Amer. J. Phys.*, vol. 42, pp. 840–846, 1974.

[22] M. S. Gupta, "Definition of instantaneous frequency and frequency measurement," *Amer. J. Phys.*, vol. 43, pp. 1087–1088, 1975.

[23] A. Papoulis, *Probability Random Variables and Stochastic Processes*. New York: McGraw-Hill, 1985, pp. 321–324.

[24] A. Papoulis, "Random modulation: A review," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-31, no. 1, pp. 96–105, 1983.

[25] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. New York: Springer Verlag, 1983, p. 23.

[26] J. J. Ebers and J. L. Moll, "Large-signal behaviour of junction transistors," *Proc. IRE*, vol. 42, pp. 1761–1772, 1954.

[27] B. Schwaderer, "Modellierung bipolarer Mikrowellentransistoren mit dem Netzwerk-Analyseprogramm SPICE," *Arch. Elek. Übertragung.*, vol. 35, no. 9, pp. 368–372, 1981.

[28] P. Gruber, *1/f-Noise Generator, Noise in Physical Systems and 1/f Noise*. New York: Elsevier Science Publishers, 1986, pp. 357–360.

[29] B. Pellegrini, F. Saletti, B. Neri and P. Terreni, *1/f Noise Generators, Noise in Physical Systems and 1/f Noise*. New York: Elsevier Science Publishers, 1986, pp. 425–428.

[30] H. Haken, *Advanced Synergetics*. Berlin, Heidelberg: Springer-Verlag, 1983, pp. 81–84.

[31] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra*. New York: Academic Press, 1974, pp. 154.

Franz X. Kaertner was born in Cham, West Germany, in 1961. He received the Dipl. Ing. degree from the Technical University of Munich, West Germany, in 1986. Since then, he has worked as a Research Assistant at the Technical University of Munich, where he is engaged in investigations of noise in nonlinear circuits.